Int. J. Heat Mass Transfer. Vol. 15, pp. 2409–2417. Pergamon Press 1972. Printed in Great Britain

A NOTE ON A MINIMUM PRINCIPLE IN BÉNARD CONVECTION

E. PALM
Department of Mechanics, University of Oslo, Blindern, Oslo 3, Norway

(*Received* 30 *June* 1971 *and in revised form* 28 *February* 1972)

Abstract—As well known, when the non-linear inertia terms in the Navier–Stoke's equations are neglected, the fluid motion is governed by the principle of minimum dissipation in the steady state. Correspondingly, in heat conduction the heat flow satisfies the principle of minimum "thermal dissipation". Considering linearized Bénard convection, a combination of these two principles is easily obtained. The aim of this note is to extend this last principle into the non-linear regime. It is assumed that the Rayleigh number is only slightly larger than the critical value, and it is then found that the minimum principle is still true.

NOMENCLATURE

a, all-over wave number;
A_n, amplitude, defined by (2.11);
g, acceleration of gravity;
h, depth of fluid layer;
i, j, k, 1, 2 or 3;
P, Prandtl number;
p, pressure;
R, Rayleigh number;
t, time;
T, temperature;
T_0, standard temperature;
u_i, velocity (u_1, u_2 horizontal velocity, u_3 vertical velocity);
x_i, space coordinate (x_1, x_2 horizontal coordinate, x_3 vertical coordinate);
α, coefficient of expansion;
δ_{ij}, Kronecker delta;
κ, thermal diffusivity;
ν, kinematic viscosity;
ρ, density;
ρ_0, standard density;
θ, temperature;
∇, vector gradient;
∇^2, Laplacian.

The superscripts are defined by (3.2)–(3.5).

1. INTRODUCTION

It is well known that for heat conduction the temperature distribution is governed by a minimum principle. This principle is easily derived from the equation of heat conduction which may be written

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T \tag{1.1}$$

where T is the temperature, t the time, κ the thermal diffusivity and ∇^2 the Laplacian. We multiply (1.1) with $\partial T/\partial t$ and integrate over the entire fluid layer (or, more general, the entire body). The right hand side is transformed by applying the Gaussian Theorem. Assuming that at the boundaries either T is time-independent or the derivative of T along the boundary normal is zero, we obtain

$$\frac{\partial}{\partial t}\tfrac{1}{2}\kappa\langle(\nabla T)^2\rangle = -\left\langle\left(\frac{\partial T}{\partial t}\right)^2\right\rangle \tag{1.2}$$

where $\langle\ \rangle$ denotes integration over the fluid layer. The quantity $\frac{1}{2}\kappa\langle(\nabla T)^2\rangle$ may properly be called the (rate of) thermal dissipation. (1.2) shows that the thermal dissipation ever decreases until the heat flow becomes steady.

Therefore, in the steady state, the thermal dissipation attains a minimum value. For heat conduction this principle is closely related to the principle of minimum entropy production as put forward by Prigogine and his collaborators (see e.g. [1]). Their principle is obtained by multiplying (1.1) with $-\partial T^{-1}/\partial t$ and integrating over the fluid layer, applying the boundary conditions. We then find that

$$\tfrac{1}{2}\kappa\left\langle T^2\frac{\partial}{\partial t}(\nabla T^{-1})^2\right\rangle = -\frac{1}{T^2}\left(\frac{\partial T}{\partial t}\right)^2. \quad (1.3)$$

Assuming that the temperature deviates only slightly from the temperature in the steady state, (1.3) may be written, approximately,

$$\frac{\partial}{\partial t}\tfrac{1}{2}\kappa\left\langle\left(\frac{\nabla T}{T}\right)^2\right\rangle = -\frac{1}{T^2}\left(\frac{\partial T}{\partial t}\right)^2 \quad (1.4)$$

The left-hand side is the time derivative of the entropy production (apart from a constant factor). It thus follows that the entropy production attains a minimum value in the steady state. It is noted that (1.4) is only true when the temperature is close to the steady state whereas (1.2) is free from such restrictions.

A similar principle as (1.2) is also valid for the motion in an incompressible fluid, disregarding the gravity and the non-linear inertia terms. The equations of motion may then be written

$$\frac{\partial u_i}{\partial t} = \nu\nabla^2 u_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} \quad (1.5)$$

where u_i is the velocity in the i-direction ($i = 1, 2, 3$), ν is the kinematic viscosity, ρ the density and p the pressure. Multiplying (1.5) with $\partial u_i/\partial t$, integrating over the entire fluid layer, applying the boundary conditions, and taking account of the incompressibility condition, leads to

$$\frac{\partial}{\partial t}\tfrac{1}{2}\nu\,\langle(\nabla u_i)^2\rangle = -\left\langle\left(\frac{\partial u_i}{\partial t}\right)^2\right\rangle \quad (1.6)$$

where we have used the summation convention. Here $\frac{1}{2}\nu\langle(\nabla u_i)^2\rangle$ is the (rate of) dissipation. According to (1.6) the dissipation ever decreases until the motion becomes steady. We thus find that the dissipation attains a minimum value in the steady state, a result which was already derived by Rayleigh ([2], p. 618). The corresponding principle concerning minimum entropy production is obtained by multiplying (1.6) with T^{-1}. Assuming that the temperature is approximately constant, T may be put inside the space integral. The left-hand side then denotes the time derivative of the entropy production which is thus seen to obtain a minimum value in the steady state.

An extension and combination of the two principles (1.2) and (1.6) is easily derived for the case of heat convection. As above we disregard the non-linear terms. Furthermore, we shall apply the Boussinesq approximation. Introducing dimensionless quantities the linearized versions of the equation of heat and the equations of motion may then be written (see section 2)

$$\frac{\partial\theta}{\partial t} = \nabla^2\theta + Ru_3 \quad (1.7)$$

$$\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} + P\theta\delta_{i3} + P\nabla^2 u_i. \quad (1.8)$$

Here R is the Rayleigh number and P the Prandtl number (defined in section 2) and δ_{i3} the Kronecker delta. Subscript 3 denotes the vertical direction. The equation of continuity takes the form

$$\frac{\partial u_i}{\partial x_i} = 0. \quad (1.9)$$

We multiply (1.7) with $P\partial\theta/\partial t$ and (1.8) with $R\partial u_i/\partial t$, integrate the equations over the entire fluid layer and add the equations.

Applying (1.9) and the Gaussian Theorem, utilizing the boundary conditions $u_i = 0$ and T is time-independent (or the derivative of T along the boundary normal is zero), we obtain

$$\frac{\partial V}{\partial t} = -\left\langle R\left(\frac{\partial u_i}{\partial t}\right)^2 + P\left(\frac{\partial T}{\partial t}\right)^2\right\rangle \quad (1.10)$$

where

$$V = P(\langle\tfrac{1}{2}(\nabla T)^2\rangle + \tfrac{1}{2}R\langle(\nabla u_i)^2\rangle - R\langle Tu_3\rangle). \tag{1.11}$$

The functional V is composed of three terms, the two first being proportional to the thermal dissipation and the dissipation, respectively. The last term is proportional to the conversion of potential energy into other energy forms. V may be called the generalized dissipation and (1.10) states that the generalized dissipation ever decreases until the motion becomes steady. In the steady state the generalized dissipation therefore attains a minimum value. This principle is closely connected to the principle on minimum entropy production for linear convection [1] which is derived by assuming that the velocity and temperature field are close to that of the steady state. We shall, however, not go in any details here.

It may perhaps be worth while to mention that the minimum principles (1.2), (1.6) and (1.10) readily follow from a somewhat more general principle. We consider the equation

$$\frac{\partial\psi}{\partial t} + L\psi = 0 \tag{1.12}$$

where ψ may be a vector and L a linear matrix operator. The problem is assumed to be self-adjoint such that

$$\left\langle\frac{\partial\psi}{\partial t}L\psi\right\rangle = \left\langle\psi L\frac{\partial\psi}{\partial t}\right\rangle.$$

Multiplying (1.12) with $\partial\psi/\partial t$ and integrating we then obtain

$$\frac{\partial}{\partial t}\left\langle\psi L\psi\right\rangle = -2\left\langle\frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial t}\right\rangle \tag{1.13}$$

which shows that the quantity $\langle\psi L\psi\rangle$ has a minimum value in the steady state.

(1.10) was derived by cancelling convection of momentum and heat. An important question is if a principle of the form (1.10) also exists if the non-linear terms are taken into account. To answer this question we note that the existence of such a principle involves that the motion approaches a steady state. This cannot be the case when the motion for example is turbulent. Further, for laminar motion we know that for sufficiently high Rayleigh numbers the motion becomes oscillatory. So obviously, if a principle of the form (1.10) exists in the non-linear regime, the corresponding Rayleigh numbers must be relatively moderate. In the next sections we shall attempt to take into account these non-linear terms, assuming that the Rayleigh number is just slightly above the critical value. It will turn out that by a minor redefinition of V, (1.10) is still valid.

2. THE BASIC EQUATIONS AND BOUNDARY CONDITIONS

We shall consider a fluid layer of infinite horizontal extent and bounded by two horizontal boundaries. For simplicity we apply the Boussinesq approximation and disregard any effect due to the material properties being temperature dependent. The equations of motion and continuity may then be written

$$\frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} - \frac{\rho g}{\rho_0}\delta_{i3} + \nu\nabla^2 u_i \tag{2.1}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2.2}$$

where we have used the summation convention, and i, k may be 1, 2, 3. x_1, x_2 are horizontal coordinates, x_3 the vertical coordinate measured positive upwards, t the time, u_i the velocity, ρ the density, ρ_0 a standard density, p the pressure, g the acceleration of gravity, ν the kinematic viscosity, δ_{ij} the Kronecker delta and ∇^2 is the Laplacian. Furthermore, the heat equation is

$$\frac{\partial T}{\partial t} + u_k\frac{\partial T}{\partial x_k} = \kappa\nabla^2 T \tag{2.3}$$

and the equation of state may be written

$$\rho = \rho_0(1 - \alpha(T - T_0)). \tag{2.4}$$

Here T denotes the temperature, κ the thermal diffusivity, α the coefficient of expansion and T_0 is a standard temperature.

The temperature may be written

$$T = T_0 - \beta x_3 + \theta \qquad (2.5)$$

where $\beta = \Delta T/h$ with ΔT denoting the difference in temperature between the lower and upper boundary and h the depth of the layer.

To get a dimensionless form of the equations we set

$$x_i = hx'_i, \qquad u_i = \kappa u'_i/h, \quad t = h^2 t'/\kappa$$
$$\theta = \kappa\nu\theta'/\alpha g h^3, \quad p = \kappa^2 \rho_0 p'/h^2.$$

Disregarding the static pressure, applying (2.4) and (2.5), and dropping the primes, we obtain

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{\partial p}{\partial x_i} + P\theta\delta_{i3} + P\nabla^2 u_i \qquad (2.6)$$

$$\frac{\partial \theta}{\partial t} + u_k \frac{\partial \theta}{\partial x_k} = \nabla^2\theta + Ru_3 \qquad (2.7)$$

$$\frac{\partial u_i}{\partial x_i} = 0. \qquad (2.8)$$

Here P is the Prandtl number and R the Rayleigh number:

$$P = \frac{\nu}{\kappa} \qquad R = \frac{\alpha g \beta h^4}{\nu\kappa}. \qquad (2.9)$$

The horizontal boundaries may be either rigid or free. In the first case $u_i = 0$ at the boundary; in the last case the vertical velocity and the shearing stresses are zero at the boundary. It will furthermore be assumed that the horizontal boundaries are either perfect heat conductors or perfect heat insulators. Applying (2.8) we then have

$$u_1 = 0, \qquad \theta = 0 \text{ or } \frac{\partial \theta}{\partial x_3} = 0 \text{ at rigid boundaries}$$

$$u_3 = \frac{\partial^2 u_3}{\partial x_3^2} = 0, \quad \theta = 0 \text{ or } \frac{\partial \theta}{\partial x_3} = 0 \text{ at free boundaries.}$$

The linearized version of equations (2.6)–(2.8) together with the proper boundary conditions lead to an eigenvalue problem. It is easily shown (and well known) that this problem is self-adjoint. Introducing that u_i and θ are proportional to $\exp \sigma t$, we therefore will find that σ is real. The onset of convection thus takes place for $\sigma = 0$ which corresponds to neglecting the operator $\partial/\partial t$ in the linearized equations. Let the corresponding value of R be denoted by $R^{(0)}$. Eliminating u_1 and u_2 and applying (2.8), the linearized equations take then the form

$$\begin{aligned} \nabla^4 u_3 + \nabla_2^2\theta &= 0 \\ R^{(0)}u_3 + \nabla^2\theta &= 0 \end{aligned} \qquad (2.10)$$

where ∇_2^2 is the two-dimensional Laplacian.

The general solution of (2.10) may be written

$$\begin{aligned} u_3 &= f(x_3)\sum A_n \cos a(x_1 \cos\phi_n + x_2 \sin\phi_n + \psi_n) \\ \theta &= g(x_3)\sum A_n \cos a(x_1 \cos\phi_n + x_2 \sin\phi_n + \psi_n) \end{aligned} \qquad (2.11)$$

where $f(x_3)$ and $g(x_3)$ satisfy the differential equations

$$\begin{aligned} \left(\frac{d^2}{dx_3^2} - a^2\right)^2 f(x_3) - a^2 g(x_3) &= 0 \\ R^{(0)} f(x_3) + \left(\frac{d^2}{dx_3^2} - a^2\right) g(x_3) &= 0. \end{aligned} \qquad (2.12)$$

3. DERIVATION OF THE MINIMUM PRINCIPLE

We multiply (2.6) with $R\, \partial u_i/\partial t$ and (2.7) with $P\, \partial\theta/\partial t$, add the equations and integrate over the entire fluid layer. Applying the Gaussian Theorem and the boundary conditions, we find

$$\begin{aligned} \frac{\partial}{\partial t} P[\tfrac{1}{2}\langle(\nabla\theta)^2\rangle + \tfrac{1}{2}R\langle(\nabla u_i)^2\rangle - R\langle\theta u_3\rangle] \\ = -\left\langle R\left(\frac{\partial u_i}{\partial t}\right)^2 + P\left(\frac{\partial\theta}{\partial t}\right)^2\right\rangle \\ - R\left\langle u_k \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial t}\right\rangle - P\left\langle u_k \frac{\partial\theta}{\partial x_k}\frac{\partial\theta}{\partial t}\right\rangle. \end{aligned} \qquad (3.1)$$

(3.1) is identical to (1.10), apart from the two last terms on the right-hand side which are due to non-linear convection of heat and momentum.

The mathematical problem now consists in recasting these two terms into an appropriate form. Since it is assumed that the Rayleigh number is only slightly above the critical value, we shall apply the perturbation method.

An arbitrary initial motion consists of an infinite number of modes. All of these, except the critical mode, decays exponentially with time. The latter, however, grows exponentially with time until the non-linear terms become important (for a more complete discussion, see [3]). We shall here only be interested in the development of the motion from that stage on, disregarding the decaying modes.

A consistent expansion scheme is

$$R = R^{(0)} + \epsilon^2 R^{(2)} + \dots \tag{3.2}$$

$$u_i = \epsilon u_i^{(1)} + \epsilon^2 u_i^{(2)} + \dots \tag{3.3}$$

$$\theta = \epsilon\theta^{(1)} + \epsilon^2\theta^{(2)} + \dots \tag{3.4}$$

$$p = \epsilon p^{(1)} + \epsilon^2 p^{(2)} + \dots \tag{3.5}$$

with

$$\frac{\partial}{\partial t} = \epsilon^2 \frac{\partial}{\partial \tau} + \dots$$

where $\partial/\partial\tau$ is of order unity. In (3.2) $R^{(1)}$ is cancelled to obtain an expansion which is also valid for large values of time. Furthermore, since the critical mode is the solution of a self adjoint problem, $\partial/\partial t$ is small. It is readily shown that $\partial/\partial t$ is (at most) of order ϵ^2; compare the remarks to (3.10). Introducing the expressions above in (2.6) and (2.7), we obtain for the first order terms

$$-\frac{\partial p^{(1)}}{\partial x_i} + P\theta^{(1)}\delta_{i3} + P\nabla^2 u_i^{(1)} = 0 \tag{3.6}$$

$$\nabla^2\theta^{(1)} + R^{(0)}u_3^{(1)} = 0. \tag{3.7}$$

For the second order terms we find

$$-\frac{\partial p^{(2)}}{\partial x_i} + P\theta^{(2)}\delta_{i3} + P\nabla^2 u_i^{(2)} = u_k^{(1)}\frac{\partial u_i^{(1)}}{\partial x_k} \tag{3.8}$$

$$\nabla^2\theta^{(2)} + R^{(0)}u_3^{(2)} = u_k^{(1)}\frac{\partial\theta^{(1)}}{\theta x_k}. \tag{3.9}$$

If we multiply (3.6) with $R^{(0)}$ and (3.7) with P, the system of equations (3.6), (3.7) constitute together with the boundary conditions a self adjoint problem (which was exploited above in cancelling the $\partial/\partial t$ operator).

(3.8) and (3.9) compose a system of inhomogeneous equations which formally has the same left hand side as the system (3.6), (3.7). In order to secure that this inhomogeneous problem possesses a solution, it must fulfil the solvability condition. Let $u_i^{(1)'}$ and $\theta^{(1)'}$ denote an arbitrary solution of the first order equations, which fulfil the boundary conditions and (2.8). We multiply (3.8) with $R^{(0)}u_i^{(1)'}$ and (3.9) with $P\theta^{(1)'}$, add the equations and integrate over the entire fluid layer, applying the boundary conditions. The solvability condition is then obtained by putting the sum of the terms on the right hand sides equal to zero. Putting first $u_i^{(1)'} = u_i^{(1)}$ and $\theta^{(1)'} = \theta^{(1)}$ we must have

$$R^{(0)}\left\langle u_k^{(1)}\frac{\partial u_i^{(1)}}{\partial x_k}u_i^{(1)}\right\rangle + P\left\langle u_k^{(1)}\frac{\partial\theta^{(1)}}{\partial x_k}\theta^{(1)}\right\rangle = 0. \tag{3.10}$$

Applying the equations of continuity and boundary conditions we notice that this relation is identically fulfilled, indicating that so far the expansion scheme is consistent. It may be noted that if $\partial/\partial t$ was assumed to be $O(\epsilon)$ (instead of ϵ^2), (3.10) would contain an additional $\partial/\partial t$-term which is inconsistent with the solvability condition.

Correspondingly the sum of the left-hand terms equal to zero gives

$$R^{(0)}\langle\theta^{(2)}u_3^{(1)}\rangle + R^{(0)}\langle\theta^{(1)}u_3^{(2)}\rangle - R^{(0)}\langle\nabla u_i^{(2)}\nabla u_i^{(1)}\rangle + \langle\nabla\theta^{(2)}\nabla\theta^{(1)}\rangle = 0. \tag{3.11}$$

Furthermore, putting

$$u_i^{(1)'} = \frac{\partial u_i^{(1)}}{\partial t} \quad\text{and}\quad \theta^{(1)'} = \frac{\partial\theta^{(1)}}{\partial t}$$

we obtain

$$R^{(0)}\left\langle u_k^{(1)}\frac{\partial u_i^{(1)}}{\partial x_k}\frac{\partial u_i^{(1)}}{\partial x_k}\right\rangle + P\left\langle u_k^{(1)}\frac{\partial \theta^{(1)}}{\partial x_k}\frac{\partial \theta^{(1)}}{\partial t}\right\rangle = 0. \quad (3.12)$$

Returning to (3.1) we note that according to (3.11) the sum of the fifth order terms ($\partial/\partial t$ is of second order) on the left-hand side is zero, and that according to (3.12) the sum of the fifth-order terms on the right-hand side is zero. Therefore, to obtain a result of interest we must in (3.1) take into account sixth-order terms.

As to the last two terms in (3.1) we first note that (see [4])

$$\left\langle u_k^{(2)}\frac{\partial u_i^{(1)}}{\partial x_k}\frac{\partial u_i^{(1)}}{\partial t}\right\rangle = \left\langle u_k^{(2)}\frac{\partial \theta^{(1)}}{\partial x_k}\frac{\partial \theta^{(1)}}{\partial t}\right\rangle = 0. \quad (3.13)$$

The two terms also give rise to other sixth-order terms which sum after some algebra (see the appendix) may be written

$$R^{(0)}\left[\left\langle u_k^{(1)}\frac{\partial u_i^{(2)}}{\partial x_k}\frac{\partial u_i^{(1)}}{\partial t}\right\rangle + \left\langle u_k^{(1)}\frac{\partial u_i^{(1)}}{\partial x_k}\frac{\partial u_i^{(2)}}{\partial t}\right\rangle\right] + P\left[\left\langle u_k^{(1)}\frac{\partial \theta^{(2)}}{\partial x_k}\frac{\partial \theta^{(1)}}{\partial t}\right\rangle + \left\langle u_k^{(1)}\frac{\partial \theta^{(1)}}{\partial x_k}\frac{\partial \theta^{(2)}}{\partial t}\right\rangle\right] = \tfrac{1}{2}P\frac{\partial}{\partial t}\{R^{(0)}\langle\theta^{(2)}u_3^{(2)}\rangle - \tfrac{1}{2}R^{(0)}\langle(\nabla u_i^{(2)})^2\rangle - \tfrac{1}{2}\langle(\nabla\theta^{(2)})^2\rangle\}. \quad (3.14)$$

The right-hand side in (3.14) is similar to the left-hand side in (3.1), except that in (3.14) R is replaced by $R^{(0)}$. Since $R^{(1)}$ is zero, $\Delta R = R - R^{(0)}$ is of second order. Furthermore, from (3.6) and (3.7) we deduce that

$$\langle\theta^{(1)}u_3^{(1)}\rangle = \langle(\nabla u_i^{(1)})^2\rangle \quad (3.15)$$

which expresses the balance between the conversion of potential energy and the rate of dissipation. Applying this, the right-hand side in (3.14) may be written

$$\tfrac{1}{2}P\frac{\partial}{\partial t}\{\tfrac{1}{2}\langle(\nabla\theta)^2\rangle + \tfrac{1}{2}R\langle(\nabla u_i)^2\rangle + \tfrac{1}{2}\Delta R\langle(\nabla u_i)^2\rangle - R\langle\theta u_3\rangle\} \quad (3.16)$$

valid to the sixth order. Introducing this result in (3.1) we end up with

$$\frac{\partial V}{\partial t} = -\left\langle R\left(\frac{\partial u_i}{\partial t}\right)^2 + P\left(\frac{\partial \theta}{\partial t}\right)^2\right\rangle \quad (3.17)$$

valid to the sixth order, where V is defined by

$$V = P\langle\tfrac{1}{2}(\nabla\theta)^2 + \tfrac{1}{2}R^{(0)}(\nabla u_1)^2 - R\theta u_3\rangle. \quad (3.18)$$

Comparing this definition of V with that given in (1.11) we note that the difference is that in (3.18) $R^{(0)}$ appears as a factor in the dissipation term instead of R. Calling V defined by (3.18) the generalized dissipation we thus have that the generalized dissipation ever decreases until the motion becomes steady. In the steady state the generalized dissipation therefore attains a minimum value.

4. DISCUSSION OF THE MINIMUM PRINCIPLE

We multiply (2.6) and (2.7) with δu_i and $\delta\theta$, respectively (instead of $\partial u_i/\partial t$ and $\partial\theta/\partial t$) where δu_i satisfies (2.8) and δu and $\delta\theta$ are given by (compare (2.11))

$$\begin{aligned}\delta u_3 &= f(x_3)\sum\delta A_n\cos a(x_1\cos\phi_n + x_2\sin\phi_n + \psi_n)\\ \delta\theta &= g(x_3)\sum\delta A_n\cos a(x_1\cos\phi_n + x_2\sin\phi_n + \psi_n).\end{aligned} \quad (4.1)$$

A procedure similar to that given above then leads to

$$\delta V = -\left\langle R\frac{\partial u_i}{\partial t}\delta u_i + P\frac{\partial\theta}{\partial t}\delta\theta\right\rangle \quad (4.2)$$

instead of (3.17). Introducing (2.11) and (4.1) and utilizing that

$$a^2u_i^{(1)} = \frac{\partial^2 u_3^{(1)}}{\partial x_3\partial x_i} \quad (i = 1, 2) \quad (4.3)$$

(4.2) takes the form

$$\frac{\partial V}{\partial A_n}\delta A_n = -\lambda\dot{A}_n\delta A_n \quad (4.4)$$

with

$$\lambda = \tfrac{1}{2}\{R\langle a^{-2}f'^2 + f^2\rangle + P\langle g^2\rangle\}.$$

From (4.4) it follows that

$$\lambda \dot{A}_n = -\frac{\partial V}{\partial A_n} \tag{4.5}$$

which shows that, to the order considered, the non-linear amplitude equations may be derived from the minimum principle. The existence of an extremum principle for the amplitude equations was derived some years ago by Busse [5] by a different approach than that given here and without giving physical interpretation of the functional which is minimized.

The minimum principle in the form (4.5) shows that the vector dA_n is directed normal to the surface $V =$ constant and such that dA_n points towards decreasing values of V. In the simple case of only two amplitudes A_1 and A_2, this fact may be exploited to solving the corresponding time dependent non-linear amplitude equations graphically by first constructing V as a function of A_1 and A_2 and then drawing curves normal to the isolines of V.

It has been suggested by Malkus [6, 7] that the physical realized motion is the one which corresponds to maximum convective heat transport. Seemingly, this suggestion is in contradiction to the minimum principle derived above. This is, however, not the case. In Malkus principle which, to the same order as considered here, was shown to be true by Schlüter *et al.* [4], the realized steady motion is only compared to other possible steady state solutions—in contrast to the present principle where the class of comparable functions are much wider. It was shown by Busse [5] that by only comparing possible steady state solutions, not only the convective heat transport, but any physical quantity described as an average property of the stationary solution is maximum for the realized steady motion at a given Rayleigh number. This conclusion by Busse is readily obtained by applying the energy equations for steady motion. These are found from (2.6) and (2.7) to be

$$\langle \theta u_3 \rangle - \langle (\nabla u_i)^2 \rangle = 0 \tag{4.6}$$

$$R\langle \theta u_3 \rangle - \langle (\nabla \theta)^2 \rangle = 0. \tag{4.7}$$

Multiplying (4.6) by R and adding the equations we obtain

$$2R\langle \theta u_3 \rangle - R\langle (\nabla u_i)^2 \rangle - \langle (\nabla \theta)^2 \rangle = 0. \tag{4.8}$$

Comparing this equation with (3.18) we find that for the steady case

$$V = \tfrac{1}{2}\Delta R \langle (\nabla u_i^{(1)})^2 \rangle. \tag{4.9}$$

For given R we thus conclude that by limiting us to only comparing the various steady state solutions, the realized solution corresponds to maximum dissipation. Applying (4.6) we note that also the convective heat transport is maximum (Malkus' principle). From (4.7) it follows that also $\langle (\nabla \theta)^2 \rangle$ is maximum, and furthermore, the various combinations of these quantities are maximum.

This unambiguity disappears when the broader class of comparable functions (2.11) are considered which lead to the minimum principle (3.17).

ACKNOWLEDGEMENT

The author is indebted to Mr. Torbjørn Ellingsen for valuable discussions.

REFERENCES

1. P. Glansdorff and I. Prigogine. On a general evolution criterion in macroscopic physics, *Physica* **30**, 351 (1964).
2. H. Lamb, *Hydrodynamics*, Sixth Ed. Dover, New York (1932).
3. W. Eckhaus, *Studies in Non-linear Stability Theory*. Springer-Verlag, Berlin (1965).
4. A. Schlüter, D. Lortz and F. Busse, On the stability of steady finite amplitude convection, *J. Fluid Mech.* **23**, 129 (1965).
5. F. H. Busse. The stability of finite amplitude cellular convection and its relation to an extremum principle, *J. Fluid Mech.* **30**, 625 (1967).
6. W. V. R. Malkus, Discrete transitions in turbulent convections, *Proc. R. Soc.* **225A**, 185 (1954).
7. W. V. R. Malkus. The heat transport and spectrum of thermal turbulence, *Proc. R. Soc.* **225A**, 196 (1954).

APPENDIX

To prove (3.14) we first consider the terms due to the non-linear heat convection. Applying the identity

$$\frac{\partial}{\partial t}\left\langle u_k^{(1)} \frac{\partial \theta^{(1)}}{\partial x_j} \theta^{(2)} \right\rangle = \left\langle \frac{\partial u_k^{(1)}}{\partial t} \frac{\partial \theta^{(1)}}{\partial x_k} \theta^{(2)} \right\rangle + \left\langle u_k^{(1)} \frac{\partial^2 \theta^{(1)}}{\partial x_k \partial t} \theta^{(2)} \right\rangle + \left\langle u_k^{(1)} \frac{\partial \theta^{(1)}}{\partial x_k} \frac{\partial \theta^{(2)}}{\partial t} \right\rangle \tag{A.1}$$

the boundary conditions and (2.8) we find that

$$\left\langle u_k^{(1)}\frac{\partial\theta^{(2)}}{\partial x_k}\frac{\partial\theta^{(1)}}{\partial t}\right\rangle = \left\langle u_k^{(1)}\frac{\partial\theta^{(1)}}{\partial x_k}\frac{\partial\theta^{(2)}}{\partial t}\right\rangle - \frac{\partial}{\partial t}\left\langle u_k^{(1)}\frac{\partial\theta^{(1)}}{\partial x_k}\theta^{(2)}\right\rangle + \left\langle\frac{\partial u_k^{(1)}}{\partial t}\frac{\partial\theta^{(2)}}{\partial x_k}\theta^{(2)}\right\rangle. \quad (A.2)$$

We shall next prove that

$$\left\langle\frac{\partial u_k^{(1)}}{\partial t}\frac{\partial\theta^{(1)}}{\partial x_k}\theta^{(2)}\right\rangle = \left\langle u_k^{(1)}\frac{\partial^2\theta^{(1)}}{\partial t\partial x_k}\theta^{(2)}\right\rangle. \quad (A.3)$$

We note that

$$a^2u_1^{(1)} = \frac{\partial^2 u_3^{(1)}}{\partial x_1\partial x_3}, \qquad a^2u_2^{(1)} = \frac{\partial^2 u_3^{(1)}}{\partial x_2 x_3}. \quad (A.4)$$

Let us first consider $k = 1$ in (A.3). Introducing (2.11) we find that

$$\frac{\partial u_1^{(1)}}{\partial t}\frac{\partial\theta^{(1)}}{\partial x_1} = gf'\sum_{m,n}\dot{A}_nA_m\cos\phi_n\cos\phi_m\sin a(x_1\cos\phi_n + x_2\sin\phi_n + \psi_n)\sin a(x_1\cos\phi_m + x_2\sin\phi_m + \psi_m). \quad (A.5)$$

Correspondingly, we derive that

$$u_1^{(1)}\frac{\partial^2\theta^{(1)}}{\partial x_1\partial t} = gf'\sum_{m,n}\dot{A}_mA_n\cos\phi_m\cos\phi_n\sin a(x_1\cos\phi_m + x_2\sin\phi_m + \psi_m)\sin a(x_1\cos\phi_n + x_2\sin\phi_n + \psi_n). \quad (A.6)$$

Changing the summation notations m and n we note that the two right-hand sides in (A.5) and (A.6) are equal and (A.3) is fulfilled for $k = 1$. The proof for $k = 2, 3$ is quite similar. Applying the boundary conditions (A.3) may be rewritten as

$$\left\langle\frac{\partial u_k^{(1)}}{\partial t}\frac{\partial\theta^{(1)}}{\partial x_k}\theta^{(2)}\right\rangle = -\left\langle u_k^{(1)}\frac{\partial\theta^{(2)}}{\partial x_k}\frac{\partial\theta^{(1)}}{\partial t}\right\rangle. \quad (A.7)$$

By means of (A.2) and (A.7) we obtain

$$\left\langle u_k^{(1)}\frac{\partial\theta^{(2)}}{\partial x_k}\frac{\partial\theta^{(1)}}{\partial t}\right\rangle + \left\langle u_k^{(1)}\frac{\partial\theta^{(1)}}{\partial x_k}\frac{\partial\theta^{(2)}}{\partial t}\right\rangle = \frac{3}{2}\left\langle u_k^{(1)}\frac{\partial\theta^{(1)}}{\partial x_k}\frac{\partial\theta^{(2)}}{\partial t}\right\rangle - \frac{1}{2}\frac{\partial}{\partial t}\left\langle u_k^{(1)}\frac{\partial\theta^{(1)}}{\partial x_k}\theta^{(2)}\right\rangle. \quad (A.8)$$

The terms due to the non-linear convection of momentum are easily found by a quite similar procedure. This leads to

$$\left[\left\langle u_k^{(1)}\frac{\partial u_i^{(2)}}{\partial x_k}\frac{\partial u_i^{(1)}}{\partial t}\right\rangle + \left\langle u_k^{(1)}\frac{\partial u_i^{(1)}}{\partial x_k}\frac{\partial u_i^{(2)}}{\partial t}\right\rangle\right] = \frac{3}{2}\left\langle u_k^{(1)}\frac{\partial u_i^{(1)}}{\partial x_k}\frac{\partial u_i^{(2)}}{\partial t}\right\rangle - \frac{1}{2}\frac{\partial}{\partial t}\left\langle u_k^{(1)}\frac{\partial u_i^{(1)}}{\partial x_k}u_i^{(2)}\right\rangle. \quad (A.9)$$

To derive (3.14) we multiply (A.8) with P and (A.9) with $R^{(0)}$ and add the expressions. The right-hand side thus obtained is transformed to the right-hand side in (3.14) by the following procedure. We multiply (3.8) with $R^{(0)}\,\partial u_i^{(2)}/\partial t$ and (3.9) with $P\,\partial\theta^{(2)}/\partial t$, add the equations and integrate over the entire fluid layer, applying the boundary conditions. (3.8) is then multiplied with $R^{(0)}u_i^{(2)}$, and (3.9) with $P\theta^{(2)}$, the equations are added and integrated as above.

UNE NOTE SUR UN PRINCIPE DE MINIMUM DANS LA CONVECTION DE BENARD

Résumé— Il est bien connu que lorsque les termes d'inertie non linéaire dans les équations de Navier–Stokes sont négligés, le mouvement du fluide est régi par le principe de dissipation minimale dans l'état stable. De manière correspondante, dans la conduction thermique, le flux thermique satisfait le principe de "dissipation thermique" minimum.

En considérant la convection linéarisée de Bénard, on a facilement obtenu une combinaison de ces deux principes. Le but de cette note est d'étendre ce dernier principe au régime non linéaire. On suppose que le nombre de Rayleigh est seulement légèrement plus grand que la valeur critique et on trouve alors que le principe de minimum est encore vrai.

EIN MINIMUM-PRINZIP IN DER BÉNARD-KONVEKTION

Zusammenfassung—Wenn man die nichtlinearen Trägheitsglieder in den Navier–Stokes'schen Gleichungen vernachlässigt, wird die Bewegung des Fluids durch das Prinzip der minimalen Dissipation im stationären Zustand beschrieben. Entsprechend erfüllt im Falle der Wärmeleitung der Wärmestrom das Prinzip der minimalen "Wärmedissipation". Durch Linearisierung der Bénard-Konvektion erhält man leicht eine Kombination dieser zwei Prinzipien. Das Ziel dieser Arbeit ist es, das letztere Prinzip auf den nicht-linearen Bereich zu erweitern. Es wird angenommen, dass die Rayleigh-Zahl nur leicht grösser ist als der kritische Wert. Es zeigt sich dann, dass das Minimum-Prinzip noch gilt.

О ПРИНЦИПЕ МИНИМУМА В КОНВЕКЦИИ БЕНАРДА

Аннотация—Хорошо известно, что если пренебречь нелинейными членами инерции в уравнениях Навье-Стокса, движение жидкости будет определяться принципом минимальной диссипации в стационарном состоянии. Соответственно, при передаче тепла теплопроводностью тепловой поток удовлетворяет принципу минимальной «тепловой диссипации». В случае линеаризованной конвекции Бенарда легко получается комбинация этих двух принципов. Целью настоящей работы является распространение последнего принципа на нелинейный случай. Принимается, что значение числа Релея лишь слегка выше критического значения. Найдено, что принцип минимума в этом случае остаётся справедливым.